

# REPRESENTATION STABILITY FOR THE COHOMOLOGY OF ARRANGEMENTS ASSOCIATED TO ROOT SYSTEMS

CHRISTIN BIBBY

**ABSTRACT.** From a root system, one may consider the arrangement of reflecting hyperplanes, as well as its toric and elliptic analogues. The corresponding Weyl group acts on the complement of the arrangement and hence on its cohomology. We consider a sequence of linear, toric, or elliptic arrangements which arise from a family of root systems of type A, B, C, or D, and we show that the rational cohomology stabilizes as a sequence of Weyl group representations. Our techniques combine a Leray spectral sequence argument similar to that of Church in the type A case along with  $\mathrm{FI}_W$ -module theory which Wilson developed and used in the linear case. A key to the proof relies on a combinatorial description, using labelled partitions, of the poset of connected components of intersections of subvarieties in the arrangement.

## 1. INTRODUCTION

In this paper, we consider toric and elliptic analogues of complex reflection arrangements associated to the classical families of root systems of type A, B, C, or D. To such a root system, one may consider the set of reflecting hyperplanes in a complex vector space. These hyperplanes are defined by integral equations, which can then be used to define codimension-one subtori or abelian subvarieties in a complex torus or a product of complex elliptic curves, respectively. Moci [Moc08] first considered these arrangements in the toric case, where they arise naturally. In each of the linear, toric, and elliptic cases, though, the complement of the arrangement comes with a natural action of the corresponding Weyl group. This action makes the rational cohomology into a representation over the Weyl group, which is the object we study.

These arrangements arising from root systems also have interesting combinatorics. In the type A case, taking all intersections of subvarieties in the arrangement forms a lattice which is isomorphic to the partition lattice. In the other linear cases, Barcelo and Ihrig [BI99] give a combinatorial description of the intersection lattice. However, in the toric and elliptic cases, intersections of hyperplanes need not be connected and form a partially ordered set which is not necessarily a lattice. In these cases, we consider the poset consisting of connected components of intersections, and in Theorem 3.3 we give a combinatorial description akin to that of Barcelo and Ihrig. Understanding the Weyl group action on the connected components of intersections is then equivalent to understanding its action on certain types of partitions, called labelled partitions.

We have already said that we're interested in the cohomology of the complement as a representation. But more specifically, we consider the sequence of representations arising from each family of root systems. We show in Theorem 4.7 that this sequence of representations stabilizes in the sense of Church-Farb [CF13]. That is,

for  $n$  large enough, if we decompose the representations into irreducibles, the multiplicity of each irreducible representation does not depend on  $n$ . As a special case, by taking the trivial representation, the orbit space enjoys homological stability (Corollary 4.12).

In the case of the symmetric group, the complement is an ordered configuration space. Church [Chu12] showed representation stability of the rational cohomology of ordered configuration spaces using a Leray spectral sequence and the partition lattice. We generalize his method of using this spectral sequence for other types of arrangements by combining it with our combinatorics and with  $\mathrm{FI}_{\mathcal{W}}$ -module theory developed by Wilson [Wil15, Wil14]. Wilson [Wil15] also showed representation stability for each linear case.

We also give a slight improvement on Church's stable range for type A elliptic arrangements in Proposition 4.16. Recently, Hersh and Reiner [HR15] showed a better improvement for the type A linear case, and we wonder if their result or methods may also be applied to these other arrangements.

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## 2. ARRANGEMENTS

**2.1. Linear, toric, and elliptic arrangements.** In general, an arrangement is a finite set  $\mathcal{A}$  of smooth connected divisors in a smooth complex variety  $V$  which intersect like hyperplanes. A *linear arrangement* is a set of hyperplanes in a complex vector space, a *toric arrangement* is a set of codimension-one subtori in a complex torus, and an *abelian arrangement* is a set of codimension-one abelian subvarieties in a complex abelian variety. In the case of an abelian arrangement, all of our abelian varieties will be products of an elliptic curve and we call it an *elliptic arrangement*. We denote the complement of  $\mathcal{A}$  in  $V$  by  $M(\mathcal{A}) = V \setminus \bigcup_{H \in \mathcal{A}} H$ .

A *component* of an arrangement  $\mathcal{A}$  is a connected component of an intersection  $\bigcap_{H \in S} H$  for some subset  $S \subseteq \mathcal{A}$ . Note that the intersections themselves need not be connected. We say that the arrangement is *unimodular* if every intersection is connected. The set of components forms a ranked poset, ordered by reverse inclusion, with rank given by the complex codimension.

Let  $F$  be a component of a linear, toric, or elliptic arrangement  $\mathcal{A}$ . For a point  $p \in F$  not contained in any smaller components of  $\mathcal{A}$ , define an arrangement  $\mathcal{A}_F$  in the tangent space  $T_p V$  consisting of hyperplanes  $H_F := T_p H$  for all  $H \supseteq F$ . If  $V$  has complex dimension  $n$ , then  $\mathcal{A}_F$  is a central hyperplane arrangement in  $T_p V \cong \mathbb{C}^n$ . This arrangement is referred to as the *localization* of  $\mathcal{A}$  at  $F$ .

**Example 2.1.** Given an  $\ell \times n$  integer matrix, each row corresponds to a map  $X^n \rightarrow X$  where  $X$  is  $\mathbb{C}$ ,  $\mathbb{C}^\times$ , or a complex elliptic curve. Assume that each row is primitive. Then by taking  $H_i$  to be the kernel of the  $i$ 'th row in  $X^n$ , the collection  $\mathcal{A} = \{H_1, \dots, H_\ell\}$  defines a linear, toric, or elliptic arrangement (respectively). If the rows are not primitive, then we may take the arrangement to be the connected components of these kernels.

Linear arrangements are always unimodular, but toric and elliptic arrangements arising from a primitive matrix will be unimodular exactly when the matrix is unimodular.

**2.2. Arrangements from root systems.** Given a root system of type A, B, C, or D, we can consider a corresponding linear, toric or elliptic arrangement. The linear arrangement is the collection of reflecting hyperplanes in  $\mathbb{C}^n$ , which are defined by linear forms with integer coefficients. Thus, we can think of this arrangement as one arising from a matrix, and use that matrix to define the toric and elliptic analogues. Moci [Moc08] and Bergvall [Ber16] describe how these arrangements arise naturally in the toric case. Even in the elliptic case, though, these arrangements are interesting to consider because they come with a natural Weyl group action, which we will describe later. We will now give a more explicit description of the arrangements of each type.

We describe Type  $C_n$  first; the others can be thought of as subarrangements. Let  $X$  be  $\mathbb{C}$ ,  $\mathbb{C}^\times$ , or a complex elliptic curve, and denote by  $X[2]$  its set of 2-torsion points, which can be thought of as the kernel of the “times two” endomorphism on  $X$ . Note that  $\mathbb{C}$  has only one 2-torsion point, while  $\mathbb{C}^\times$  has two and an elliptic curve has four. To include all three types at once, we will write the group operation additively (but note that we are using the multiplicative structure on  $\mathbb{C}^\times$ ).

A type C root system has roots given by (i) an integer vector of length  $\sqrt{2}$  and (ii) twice an integer vector of length 1. Thus, an arrangement of type  $C_n$  is given by a matrix of the following form:

$$\begin{matrix} & i & & k & & j & & \\ \begin{pmatrix} \vdots & & & & & & & \vdots \\ 0 & \dots & 0 & 1 & 0 & \dots & \dots & 0 & -1 & 0 & \dots & 0 \\ \vdots & & & & & & & & & & & \vdots \\ 0 & \dots & 0 & 1 & 0 & \dots & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & & & & & & & \vdots \\ 0 & \dots & & \dots & 0 & 2 & 0 & \dots & & \dots & 0 \\ \vdots & & & & & & & & & & & \vdots \end{pmatrix} \end{matrix}$$

where  $1 \leq i < j \leq n$  and  $1 \leq k \leq n$ . There are three distinct types of rows, and hence three types of “hyperplanes” in our arrangement.

We denote  $H_{ij}$  and  $H'_{ij}$  to be the kernels of the first two types, respectively, which are connected. The third type has connected components indexed by the 2-torsion points  $X[2]$ ; denote them by  $H_k^z$  with  $z \in X[2]$ . That is,  $H_k^z$  is the collection of all points in  $X^n$  with the  $k$ 'th coordinate equal to  $z$ . Then the arrangement of type  $C_n$  is the collection

$$(C_n) \quad \{H_{ij}, H'_{ij}, H_k^z \mid 1 \leq i < j \leq n, 1 \leq k \leq n, z \in X[2]\}.$$

For a type  $B_n$  root system, we take integer vectors of length  $\sqrt{2}$  or length 1. This corresponds to replacing the 2's in the matrix with 1's. Thus, we have the arrangement

$$(B_n) \quad \{H_{ij}, H'_{ij}, H_k^0 \mid 1 \leq i < j \leq n, 1 \leq k \leq n\}.$$

Note that in the linear case, the type B and C arrangements are equal.

A type  $D_n$  root system has only the integer vectors of length  $\sqrt{2}$ , and so we get the arrangement

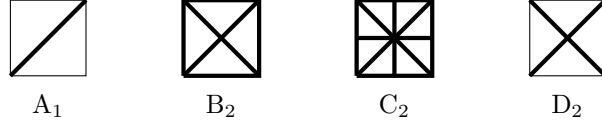
$$(D_n) \quad \{H_{ij}, H'_{ij} \mid 1 \leq i < j \leq n\}.$$

For type  $A_{n-1}$ , we consider only the first type and have

$$(A_{n-1}) \quad \{H_{ij} \mid 1 \leq i < j \leq n\}.$$

Note that toric and elliptic arrangements of types B, C, and D, are not unimodular, as  $H_{ij} \cap H'_{ij}$  has connected components indexed by  $X[2]$ . More specifically,  $H_{ij} \cap H'_{ij}$  is the collection of points whose  $i$ 'th and  $j$ 'th coordinates are both equal to each other and their negative, hence equal to a 2-torsion point.

**Example 2.2.** The best pictures we have for these arrangements are in  $n = 2$  with the real version of linear and toric arrangements. We draw here the pictures of the toric case, in  $S^1 \times S^1$ ; the subtori of the arrangement are the thickened lines.



**2.3. Weyl group action.** We claim that each of the arrangements just described comes with a natural Weyl group action, on both the poset of components and on the complement. We describe this action, starting with type C. Consider the hyperoctahedral group  $W_n = \mathbb{Z}/2 \wr S_n = (\mathbb{Z}/2)^n \rtimes S_n$ , the Weyl group for types  $B_n$  and  $C_n$ . The Weyl groups for types  $A_{n-1}$  and  $D_n$  are subgroups of  $W_n$ .

Note that  $W_n$  acts on  $X^n$  via a combination of permuting the coordinates and negating (inverting) some; the subgroup of type  $D_n$  will only negate an even number and the subgroup  $S_n$  will negate none. More specifically, given  $w = (\epsilon, \sigma) \in W_n$  and  $x = (x_1, \dots, x_n) \in X^n$ ,  $w \cdot x$  has  $\epsilon_i x_i$  in the  $\sigma(i)$ -th coordinate. Here, we consider  $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in (\mathbb{Z}/2)^n$ , with  $\mathbb{Z}/2 = \{\pm 1\}$ . This action gives us an action on our set of subvarieties:

- $w \cdot H_{ij} = H_{\sigma(i)\sigma(j)}$  if  $\epsilon_i \epsilon_j = 1$ ,
- $w \cdot H_{ij} = H'_{\sigma(i)\sigma(j)}$  if  $\epsilon_i \epsilon_j = -1$ ,
- $w \cdot H'_{ij} = H'_{\sigma(i)\sigma(j)}$  if  $\epsilon_i \epsilon_j = 1$ ,
- $w \cdot H'_{ij} = H_{\sigma(i)\sigma(j)}$  if  $\epsilon_i \epsilon_j = -1$ ,
- $w \cdot H_i^z = H_{\sigma(i)}^z$ .

Thus,  $W_n$  acts on the complements of type  $B_n$ ,  $C_n$ , and  $D_n$  arrangements, while  $S_n$  acts on the complement of the type  $A_{n-1}$  arrangements.

### 3. COMBINATORIAL DESCRIPTION OF COMPONENTS

The goal of this section is provide a combinatorial description of components, which allows to better understand and handle the Weyl group action on components. This will allow us to break down representations which appear in Section 4 into simpler building blocks, which we can then use to show stability. We start by describing the combinatorial objects needed and then show their relationship to components in Theorem 3.3.

**3.1. Labelled partitions.** Let  $S$  and  $L$  be sets. We say that a *partition of  $S$  labelled by  $L$* , or an  $L$ -labelled partition of  $S$ , is a partition  $\Sigma$  of  $S$ , together with a subset  $T \subseteq \Sigma$  and injection  $f : T \rightarrow L$ . We say that the parts in  $T$  are the *labelled parts* of  $\Sigma$ , while  $\Sigma \setminus T$  consists of the *unlabelled parts*. For  $p \in T$  corresponding to  $z \in L$ , we say that  $p$  is *labelled by  $z$*  and use the notation  $p = \Sigma_z$ . We use the convention that if  $z$  is not in the image of  $f$ , then  $\Sigma_z = \emptyset$ . One may similarly define a partition of a number  $k$  labelled by  $L$ .

Given one of our root systems, we will define a particular set of labelled partitions, which in Theorem 3.3 will be shown to describe the components of the corresponding arrangement. First, we introduce some notation. Again let  $X$  be  $\mathbb{C}$ ,  $\mathbb{C}^\times$ , or a complex elliptic curve with 2-torsion points  $X[2]$ . Let  $[n] = \{1, 2, \dots, n\}$  and  $\mathbf{n} = \{1, \bar{1}, 2, \bar{2}, \dots, n, \bar{n}\}$ . For  $S \subseteq \mathbf{n}$ , let  $\bar{S} = \{\bar{x} \mid x \in S\}$ , taking  $\bar{\bar{k}} = k$ .

Let  $\mathcal{P}_n$  be the set of partitions  $\Sigma$  of  $\mathbf{n}$  labelled by  $X[2]$  such that

- (i) for every  $S \in \Sigma$ ,  $\bar{S} \in \Sigma$ , and
- (ii)  $S = \bar{S}$  if and only if  $S$  is labelled.

Each type of root system will correspond to a subset of  $\mathcal{P}_n$  as follows: The subset of type  $C_n$  is just  $\mathcal{P}_n$  itself. For type  $B_n$ , we take the subset of all  $\Sigma \in \mathcal{P}_n$  such that if  $|\Sigma_z| = 2$  then  $z = 0$ . For type  $D_n$ , we take the subset of all  $\Sigma \in \mathcal{P}_n$  such that  $|\Sigma_z| \neq 2$  for any  $z \in X[2]$ . For type  $A_{n-1}$ , we take the subset of all  $\Sigma = \Sigma_+ \cup \bar{\Sigma}_+$  where  $\Sigma_+ \vdash [n]$ . If the type is understood, we will denote the set in consideration by  $\mathcal{C}_n$ .

In each type,  $\mathcal{C}_n$  is a partially ordered set, with  $\Sigma < \Sigma'$  if  $\Sigma$  is a refinement of  $\Sigma'$  such that  $\Sigma_z \subseteq \Sigma'_z$  for each  $z \in X[2]$ . That is, we order it by refinements which respect the labelling. Moreover,  $\mathcal{C}_n$  is a ranked poset with  $\text{rk}(\Sigma) = n - \frac{\ell}{2}$ , where  $\ell$  is the number of unlabelled parts of  $\Sigma$ .

After discussing the Weyl group action, we give an example of these posets in Example 3.1.

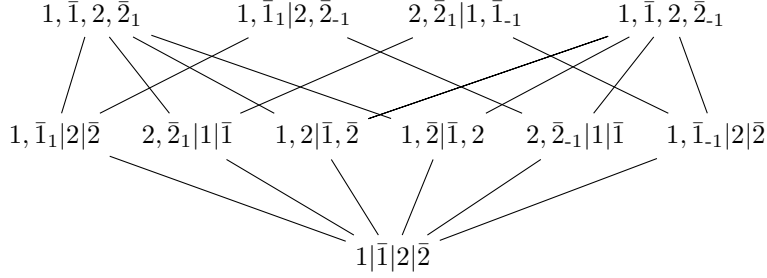
**3.2. Weyl group action.** In type A, the set  $\mathcal{C}_n$  is really just the partition lattice, which has an action of  $S_n$ . In the other types, we have an action of the hyperoctahedral group  $W_n = \mathbb{Z}_2 \wr S_n$ . This action is induced by its action on  $\mathbf{n}$ , but we will describe it more explicitly. Let  $w = (\epsilon, \sigma) \in W_n$ . Then if  $k \in [n]$ , we have  $w \cdot k = \overline{\sigma(k)}$  if  $\epsilon_k = -1$  and  $w \cdot k = \sigma(k)$  if  $\epsilon_k = 1$ . Moreover,  $w \cdot \bar{k} = \overline{w \cdot k}$ . Now, for  $S \subseteq \mathbf{n}$ , we have  $w \cdot S = \{w \cdot x \mid x \in S\}$ . Then  $w \cdot \Sigma = \{w \cdot S \mid S \in \Sigma\}$ , with labels so that  $(w \cdot \Sigma)_z = w \cdot (\Sigma_z)$ .

Given a partition  $\Sigma \in \mathcal{P}_n$ , we can define a partition  $\widehat{\Sigma}$  of  $n$  labelled by  $X[2]$ , where  $\widehat{\Sigma}_z = \frac{|\Sigma_z|}{2}$  and the unlabelled parts are given by  $|S|$  for each pair of unlabelled parts  $S, \bar{S} \in \Sigma$ . For example, if  $\Sigma$  is the labelled partition  $1, \bar{1}_0 | 2, \bar{4} | \bar{2}, 4 | 3 | \bar{3}$  of  $\mathbf{4}$ , then we have  $\widehat{\Sigma} = (1_0, 2, 1)$ , a labelled partition of  $4$ .

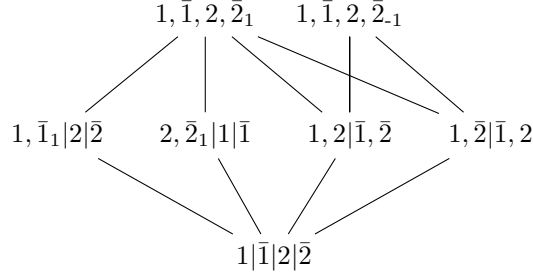
The  $W_n$ -action preserves  $\widehat{\Sigma}$ ; that is,  $\widehat{w\Sigma} = \widehat{\Sigma}$  for all  $w \in W_n$ . Moreover, this determines the orbits. If  $\mathcal{Q}_n$  is the set of partitions of  $n$  labelled by  $X[2]$ , then the orbits are given by  $\mathcal{O}_\lambda = \{\Sigma \mid \widehat{\Sigma} = \lambda\}$  with  $\lambda \in \mathcal{Q}_n$ . Note that some of these  $\mathcal{O}_\lambda$  may be empty (that is, we don't need all of  $\mathcal{Q}_n$ ), but this will not affect our results.

**Example 3.1.** Here, we draw the Hasse diagram of the poset  $\mathcal{C}_2$  in each type, taking  $X = \mathbb{C}^\times$ . Note that the subscripts on some of the parts in the partitions denote the labelling; here, our two-torsion points are  $X[2] = \{\pm 1\}$ . We start with type C, the largest poset.

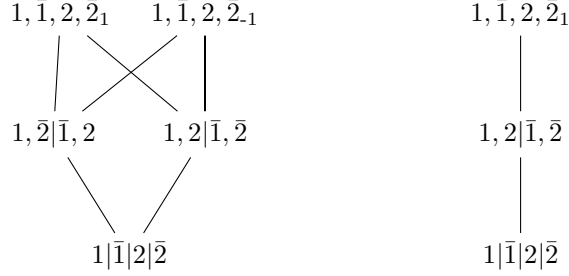
$C_2$ :



$B_2$ :



$D_2$  and  $A_1$ :

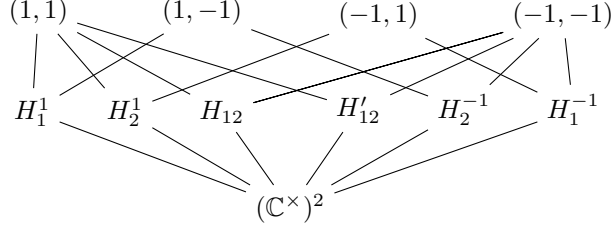


See Example 4.14 for a depiction of the orbits in the case of a  $B_2$  toric arrangement.

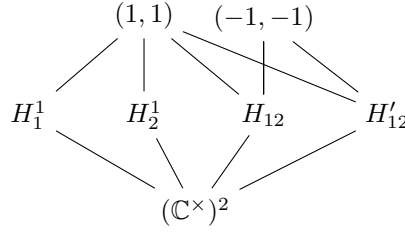
**3.3. Components as labelled partitions.** The goal of this section is to prove that these labelled partitions give a combinatorial description of components of the arrangement. This description will help us to get a handle on certain representations in Lemma 4.6. Before proving our claim in Theorem 3.3, we consider an example.

**Example 3.2.** We draw the Hasse diagrams for the poset of components in the case of  $n = 2$  and  $X = \mathbb{C}^\times$ . The bijection with the posets in Example 3.1 should be visible in these diagrams.

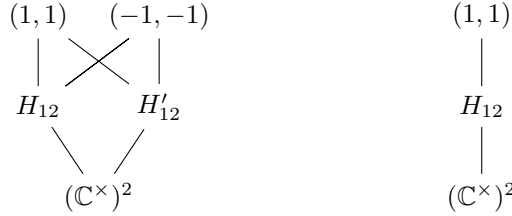
$C_2$ :



$B_2$ :



$D_2$  and  $A_1$ :



**Theorem 3.3.** *Given a root system of type  $B_n$ ,  $C_n$ , or  $D_n$ , let  $\mathcal{A}_n$  be the corresponding linear, toric, or elliptic arrangement, let  $\mathcal{C}_n$  be the corresponding set of labelled partitions, and let  $W_n$  be the hyperoctahedral group. Then there is a  $W_n$ -equivariant isomorphism of ranked posets between  $\mathcal{C}_n$  and the components of  $\mathcal{A}_n$ .*

*Proof.* We will first describe the bijection and prove the theorem for type C, and then we restrict to our other types.

*The bijection in type C:*

Let  $\Sigma \in \mathcal{P}_n$ , our poset of type C. For  $S \in \Sigma$ , take the collection of subvarieties  $H_{ij}$  if  $i, j \in S$  or  $\bar{i}, \bar{j} \in S$ ;  $H'_{ij}$  if  $i, \bar{j} \in S$  or  $\bar{i}, j \in S$ ; and  $H_i^z$  if  $S = \Sigma_z$  and  $i \in S$ . Denote the intersection of these subvarieties by  $F_S$ . If  $S$  is unlabelled, we need only consider one from its pair, since  $F_S = F_{\bar{S}}$ . First note that  $F_S$  is connected; if  $S$  is unlabelled then  $F_S \cong X^{n-|S|+1}$ , and if  $S = \Sigma_z$  then

$$F_S = \{(x_1, \dots, x_n) \in X^n \mid x_i = z \text{ for } i, \bar{i} \in S\} \cong X^{n-|S|/2}.$$

Now we take  $F_\Sigma$  to be the intersection of  $F_S$  for all  $S \in \Sigma$ . This is again connected, because if  $S$  and  $T$  are distinct parts in  $\Sigma$  with  $S \neq \bar{T}$ , we are imposing conditions on distinct sets of coordinates in  $X^n$ . Thus,  $F_\Sigma$  is a component of the arrangement  $\mathcal{A}_n$ . Moreover, if  $\Sigma$  and  $\Sigma'$  are distinct elements of  $\mathcal{P}_n$ , then  $F_\Sigma$  and  $F_{\Sigma'}$  are distinct subvarieties of  $X^n$ .

Now let  $F$  be a component of  $\mathcal{A}_n$ , and we will define a labelled partition  $\Sigma$  in  $\mathcal{P}_n$  such that  $F = F_\Sigma$ . From  $F$ , we can define an equivalence relation on  $\mathbf{n}$  as follows:  $i \sim j$  iff  $\bar{i} \sim \bar{j}$  iff  $H_{ij} \supseteq F$ ,  $i \sim \bar{j}$  iff  $\bar{i} \sim j$  iff  $H'_{ij} \supseteq F$ , and  $i \sim \bar{i}$  iff  $H_i^z \supseteq F$  for some  $z$ . This gives (by taking equivalence classes) a partition of  $\mathbf{n}$  where some parts satisfy  $S = \bar{S}$  and the others come in pairs  $(S, \bar{S})$ . We will label the self-barred parts so that we get an element of  $\mathcal{P}_n$ . If  $S = \bar{S}$ , then for each  $i \in S$ , there exists a  $z \in X[2]$  with  $H_i^z \supseteq F$ . Moreover, this  $z$  is the same no matter which  $i \in S$  we consider: if  $i$  and  $j$  are both in  $S$ , then  $i \sim j$  and  $i \sim \bar{j}$ , which means  $F$  is contained in a connected component of  $H_{ij} \cap H'_{ij}$  (the one corresponding to our  $z$ ). Thus, we may label  $S$  by  $z$ . Moreover, if  $S$  and  $T$  are distinct self-barred parts, they correspond to different elements of  $X[2]$ ; otherwise, we'd have for  $i, \bar{i} \in S$  and  $j, \bar{j} \in T$  such that  $H_{ij} \supseteq F$  and hence  $i \sim j$ .

*Compatibility with order and rank:* (Type C)

Assume that  $\Sigma$  is a refinement of  $\Sigma'$  such that  $\Sigma_z \subseteq \Sigma'_z$  for all  $z \in X[2]$ . For all  $S \in \Sigma$ , there exists  $T \in \Sigma'$  such that  $S \subseteq T$ , and all  $T \in \Sigma'$  have such an  $S$ . Moreover, if  $S = \Sigma_z$ , then  $T = \Sigma'_z$ . Since  $S \subseteq T$  implies  $F_S \supseteq F_T$ , we have  $F_\Sigma = \bigcap_{S \in \Sigma} F_S \supseteq \bigcap_{T \in \Sigma'} F_T = F_{\Sigma'}$ .

Let  $\Sigma \in \mathcal{P}_n$  with  $\ell$  unlabelled parts. In our construction of  $F_\Sigma$ , we see that the codimension of  $F_S$  is equal to  $|S| - 1$  if  $S$  is unlabelled and  $|S|/2$  if  $S$  is labelled. This means that the codimension of  $F_\Sigma$  is

$$\sum_{(S, \bar{S}) \in \Sigma} (|S| - 1) + \sum_{z \in X[2]} |\Sigma_z|/2 = n - \frac{\ell}{2}$$

which is equal to the rank of  $\Sigma$ .

*Compatibility with Weyl group action:* (Type C)

Let  $w = (\epsilon, \sigma) \in W_n$  and  $F$  a component of  $\mathcal{A}_n$ . If  $H_{ij} \supseteq F$ , then

$$wF \subseteq wH_{ij} = \begin{cases} H_{\sigma(i)\sigma(j)} & \text{if } \epsilon_i \epsilon_j = 1 \\ H'_{\sigma(i)\sigma(j)} & \text{if } \epsilon_i \epsilon_j = -1 \end{cases}$$

Similarly, if  $H'_{ij} \supseteq F$ , then

$$wF \subseteq wH'_{ij} = \begin{cases} H'_{\sigma(i)\sigma(j)} & \text{if } \epsilon_i \epsilon_j = 1 \\ H_{\sigma(i)\sigma(j)} & \text{if } \epsilon_i \epsilon_j = -1 \end{cases}$$

Finally, if  $H_i^z \supseteq F$ , then  $H_{\sigma(i)}^z \supseteq wF$ . The first two pieces imply that if  $S \in \Sigma$  is unlabelled then  $wS$  is an unlabelled part of  $w\Sigma$ , and the last one implies that  $(w\Sigma)_z = w(\Sigma_z)$ .

*Type B<sub>n</sub>:*

Let  $\mathcal{C}_n$  be the set of  $\Sigma \in \mathcal{P}_n$  such that if  $|\Sigma_z| = 2$  then  $z = 0$ . Given  $\Sigma \in \mathcal{C}_n$ , we may construct  $F_\Sigma$  as above, but we need to show that  $F_\Sigma$  is a component of the type B<sub>n</sub> arrangement. The type B<sub>n</sub> arrangement is a subarrangement of type C<sub>n</sub>, where we exclude  $H_i^z$  for  $z \neq 0$ . It is clear that if  $S \in \Sigma$  is unlabelled, then  $F_S$  is a component; we need only worry about  $F_{\Sigma_z}$ . If  $|\Sigma_z| = 2$ , then  $z = 0$ , and we have  $F_{\Sigma_z} = H_i^0$ . If  $|\Sigma_z| \neq 2$ , then consider the intersection  $H_{\Sigma_z}$  of the subvarieties  $H_{ij}$  and  $H'_{ij}$  for  $i, j \in \Sigma_z$ . This intersection is not connected, but its connected components are indexed by  $X[2]$ , and  $F_{\Sigma_z}$  is the connected component indexed by  $z$ .

We also need to show that in the inverse map, if we are restricting ourselves to components of the type B<sub>n</sub> arrangement, that the partition we get will not have



$|\Sigma_z| = 2$  for  $z \neq 0$ . Suppose that  $|\Sigma_z| = 2$ ; then there exists  $i$  such that  $i \sim \bar{i}$  but no  $j$  with  $i \sim j$  or  $i \sim \bar{j}$ . This implies that  $H_i^z \supseteq F$  but no  $H_{ij}$  or  $H'_{ij}$  contains  $F$ . The only way this can be a component in type B is if  $z = 0$ .

*Type  $D_n$ :*

Now let  $\mathcal{C}_n$  be the set of  $\Sigma \in \mathcal{P}_n$  such that  $|\Sigma_z| \neq 2$  for any  $z \in X[2]$ . Given such  $\Sigma$ , we may again construct  $F_\Sigma$  as above, but we need to show that this is a component of the type  $D_n$  arrangement. As in type B, we need only worry about  $F_{\Sigma_z}$  being a component. But since  $|\Sigma_z|$  is never 2, we will have  $F_{\Sigma_z}$  as a connected component of the intersection  $H_{\Sigma_z} = \bigcap_{i,j \in \Sigma_z} (H_{ij} \cap H'_{ij})$ .

On the other hand, suppose that we have a component  $F$  of the type D arrangement and construct the corresponding partition  $\Sigma \in \mathcal{P}_n$ . If  $\Sigma_z = \{i, \bar{i}\}$ , then there is no  $j$  such that  $H_{ij}$  or  $H'_{ij}$  contains  $F$ , contradicting the fact that  $F$  is a component.  $\square$

**Remark 3.4.** The analogous statement for type  $A_{n-1}$  is clear, because the poset  $\mathcal{C}_n$  in this case is equivalent to the partition lattice of the set  $[n]$ .

**Remark 3.5.** In the linear case, our description is equivalent to that given by Barcelo and Ihrig [BI99, Theorems 3.1&4.1]. They showed that the poset in question is also isomorphic to the lattice of parabolic subgroups of the Weyl group. It is also worth noting that in the type B/C linear case, this is the Dowling lattice. But in other cases, this labelling helps us take into account the more complicated structure of having multiple connected components of intersections.

#### 4. REPRESENTATION STABILITY

Our goal is to show representation stability for the cohomology of our arrangements, but we first briefly review representation stability and its main tool of FI $\mathcal{W}$ -modules. Throughout this section, we let  $\mathcal{W}_n$  denote either the symmetric group  $S_n$  (type A) or the hyperoctahedral group  $W_n$  (type B/C). For more details on the theory, we refer the reader to [CEF15, CF13] for the case of  $S_n$  (and much more) and [Wil14, Wil15] for the case of  $W_n$  (as well as the type D Weyl group, which we do not discuss here). Note that we are working over characteristic zero throughout.

**4.1.  $\mathcal{W}_n$ -representation stability.** To discuss representation stability for a sequence of groups, one needs a consistent way of describing the irreducible representations. There are many cases in which this can be done, including the classical families of Weyl groups.

For the symmetric group  $S_n$ , irreducible representations are indexed by partitions of  $n$ . If we consider a partition  $\lambda = (\lambda_1, \dots, \lambda_\ell)$  of  $k$  with  $\lambda_1 \geq \dots \geq \lambda_\ell > 0$  and  $n \geq \lambda_1 + k$ , we may write  $V(\lambda)_n$  to denote the irreducible representation of  $S_n$  indexed by the partition  $\lambda[n] := (n - k, \lambda_1, \dots, \lambda_\ell)$ . For example, in this notation,  $V(0)_n$  is always the trivial representation and  $V(1)_n$  is always the standard representation.

For the hyperoctahedral group  $W_n$ , irreducible representations are indexed by pairs of partitions  $\lambda = (\lambda^+, \lambda^-)$  where  $|\lambda^+| + |\lambda^-| = n$ . Given a pair of partitions  $\lambda = (\lambda^+, \lambda^-)$ , where  $\lambda^-$  is a partition of  $k$ , and  $n$  large enough, we may write  $V(\lambda)_n$  to be the irreducible representation of  $W_n$  corresponding to  $(\lambda^+[n - k], \lambda^-)$ . For example,  $V(0, 0)_n$  is always the trivial representation.

We start with a *consistent* sequence  $\{V_n\}$  of  $\mathcal{W}_n$ -representations; that is, each  $V_n$  is a  $\mathcal{W}_n$ -representation along with  $\mathcal{W}_n$ -equivariant maps  $\varphi_n : V_n \rightarrow V_{n+1}$ . Such

a sequence is said to be *uniformly representation stable* with stable range  $n \geq N$  if for  $n \geq N \dots$

- (1) the map  $\varphi_n$  is injective,
- (2) the image  $\varphi_n(V_n)$  generates  $V_{n+1}$  as a  $k[\mathcal{W}_{n+1}]$ -module, and
- (3)  $V_n = \bigoplus_{\lambda} c_{\lambda} V(\lambda)_n$ , where the multiplicities  $c_{\lambda}$  do not depend on  $n$ .

**4.2.  $\mathbf{FI}_{\mathcal{W}}$ -modules.** Consider the category  $\mathbf{FI}_{\mathcal{W}}$  (where  $\mathcal{W}$  denotes either type A or type B/C) consisting of objects  $\mathbf{n}$  (with  $\mathbf{0} = \emptyset$ ) and morphisms  $f : \mathbf{m} \rightarrow \mathbf{n}$  which are injections such that  $f(\bar{k}) = \overline{f(k)}$  for all  $k \in \mathbf{m}$ , also requiring that  $f([n]) \subseteq [n]$  if  $\mathcal{W}$  is type A. An  $\mathbf{FI}_{\mathcal{W}}$ -module is a functor  $V$  from the category  $\mathbf{FI}_{\mathcal{W}}$  to the category of  $\mathbb{Q}$ -modules. We denote by  $V_n$  the image of  $\mathbf{n}$ . Since  $\text{End}(\mathbf{n}) = \mathcal{W}_n$  in the category  $\mathbf{FI}_{\mathcal{W}}$ , the  $\mathbb{Q}$ -module  $V_n$  comes equipped with an action of  $\mathcal{W}_n$ . Moreover, the natural inclusions  $\mathbf{n} \hookrightarrow \mathbf{n} + \mathbf{1}$  induce  $\mathcal{W}_n$ -equivariant maps  $V_n \rightarrow V_{n+1}$ , making the sequence  $\{V_n\}$  a consistent sequence of  $\mathcal{W}_n$ -representations.

A map of  $\mathbf{FI}_{\mathcal{W}}$ -modules is a natural transformation. We say  $U$  is a *sub- $\mathbf{FI}_{\mathcal{W}}$ -module* of  $V$  if there is a map  $U \rightarrow V$  such that  $U_n$  is a subrepresentation of  $V_n$  for all  $n$ . An  $\mathbf{FI}_{\mathcal{W}}$ -module  $V$  is *finitely generated* if there is a finite set of elements of  $\coprod V_n$  that are not contained in any proper sub- $\mathbf{FI}_{\mathcal{W}}$ -module. An  $\mathbf{FI}_{\mathcal{W}}$ -module  $V$  has *stability degree*  $\leq s$  if  $(V_{n+a})_{\mathcal{W}_n} \cong (V_{n+1+a})_{\mathcal{W}_{n+1}}$  for every  $a \geq 0$  and  $n \geq s$ , where the subscript denotes the coinvariants. We say that  $V$  has *weight*  $\leq d$  if for all  $n$ , every irreducible representation  $V(\lambda)_n$  appearing with nonzero multiplicity in  $V_n$  satisfies  $|\lambda| \leq d$  (if  $\lambda$  is a partition) or  $|\lambda^+| + |\lambda^-| \leq d$  (if  $\lambda = (\lambda^+, \lambda^-)$  is a pair of partitions). Again, we refer the reader to [Wil14, Wil15] for more details on these concepts; we state here the main properties and example on which our results rely.

**Proposition 4.1.**

- (1) [Wil14, Prop. 4.18 & Def. 4.1] Assume that  $f : U \rightarrow V$  is a map of  $\mathbf{FI}_{\mathcal{W}}$ -modules which are finitely generated with weight  $\leq d$  and stability degree  $\leq s$ . Then  $\ker(f)$  and  $\text{coker}(f)$  are both finitely generated with weight  $\leq d$  and stability degree  $\leq s$ .
- (2) Assume that  $0 \rightarrow U \rightarrow V \rightarrow Q \rightarrow 0$  is a short exact sequence of  $\mathbf{FI}_{\mathcal{W}}$ -modules, where  $U$  and  $Q$  are both finitely generated with weight  $\leq d$  and stability degree  $\leq s$ . Then so is  $V$ .

In the above proposition, note that while Wilson explicitly stated part (1) about kernels and cokernels, a similar argument gives part (2) for extensions.

**Theorem 4.2.** Suppose that  $E_*^{pq}$  is a first quadrant spectral sequence of  $\mathbf{FI}_{\mathcal{W}}$ -modules which converges to the  $\mathbf{FI}_{\mathcal{W}}$ -module  $H^{p+q}$ . If for some  $r$ , each  $E_r^{pq}$  is finitely generated with weight  $\leq d_{p,q}$  and stability degree  $\leq s_{p,q}$ , then  $H^i$  is finitely generated with weight  $\leq \max\{d_{p,i-p}\}$  and stability degree  $\leq \max\{s_{p,i-p}\}$ .

*Proof.* Let  $r$  be such that  $E_r^{pq}$  is a finitely generated  $\mathbf{FI}_{\mathcal{W}}$ -module for each  $p$  and  $q$ , with weight  $\leq d_{p,q}$  and stability degree  $\leq s_{p,q}$ . Since  $E_{\infty}^{pq}$  is a subquotient of  $E_r^{pq}$ , we have from part (1) of Proposition 4.1 that  $E_{\infty}^{pq}$  is finitely generated with weight  $\leq d_{p,q}$  and stability degree  $\leq s_{p,q}$ .

Fix  $i$ , and consider the filtration  $F_0 \subseteq \dots \subseteq F_i = H^i$ , where  $F_j/F_{j-1} = E_{\infty}^{j,i-j}$ . Since  $F_0$  and  $F_1/F_0$  are both finitely generated, so is  $F_1$  by part (2) of Proposition 4.1, with weight  $\leq \max\{d_{0,i}, d_{1,i-1}\}$  and stability degree  $\leq \max\{s_{0,i}, s_{1,i-1}\}$ .

Repeating this for each  $F_{j-1}$  and  $F_j/F_{j-1}$  ( $j = 1, 2, \dots, i$ ) gives that  $H^i$  is finitely generated with the desired bounds on weight and stability degree.  $\square$

**Remark 4.3.** The type A case of Theorem 4.2 was stated by Kupers and Miller [KM15, Cor. 2.5] as well as by Jimenez Rolland [JR15, Thm. 3.3], but we have not seen it stated for other  $\mathrm{FI}_{\mathcal{W}}$ -modules even though it follows easily from Wilson's work. Spectral sequence arguments have also been used in specific cases; see for example [CEF15, Thm. 6.3.1] for an argument on the bounds for weight and stability degree in the case of configuration spaces.

**Theorem 4.4.** [Wil14, Thm. 4.26], [CEF15, Thm. 2.58] *If  $V$  is an  $\mathrm{FI}_{\mathcal{W}}$ -module, with  $\mathcal{W}$  of type A or B/C, which is finitely generated with weight  $\leq d$  and stability degree  $\leq s$ , then the sequence  $\{V_n\}$  of  $\mathcal{W}_n$ -representations with maps  $V_n \rightarrow V_{n+1}$  induced by the natural inclusions  $\mathbf{n} \rightarrow \mathbf{n}+1$  is uniformly representation stable with stable range  $n \geq d + s$ .*

In the following example, we describe particularly nice  $\mathrm{FI}_{\mathcal{W}}$ -modules, which will be useful for us in the next section.

**Example 4.5.** ([Wil15, Ex. 1.5.5]) Let  $U$  be a  $\mathcal{W}_k$ -representation which is finite dimensional. Consider the  $\mathrm{FI}_{\mathcal{W}}$ -module  $M_{\mathcal{W}}(U)$  which takes  $\mathbf{n}$  to 0 if  $n < k$  and otherwise to the  $\mathcal{W}_n$ -representation  $\mathrm{Ind}_{\mathcal{W}_k \times \mathcal{W}_{n-k}}^{\mathcal{W}_n} U \boxtimes \mathbb{Q}$ , where  $U \boxtimes \mathbb{Q}$  is the external tensor product of  $U$  with the trivial  $\mathcal{W}_{n-k}$ -representation  $\mathbb{Q}$ .  $M_{\mathcal{W}}(U)$  is a finitely generated  $\mathrm{FI}_{\mathcal{W}}$ -module with weight  $\leq k$  and stability degree  $\leq k$ . Thus, the sequence of induced representations

$$\left\{ \mathrm{Ind}_{\mathcal{W}_k \times \mathcal{W}_{n-k}}^{\mathcal{W}_n} U \boxtimes \mathbb{Q} \right\}$$

is representation stable with stable range  $n \geq 2k$ .

**4.3. Arrangements associated to root systems.** Let  $X$  be  $\mathbb{C}$ ,  $\mathbb{C}^\times$ , or a complex elliptic curve, and let  $\mathcal{A}_n$  be a rational, toric, or elliptic arrangement (resp.) of type  $B_n$ ,  $C_n$ , or  $D_n$ , with complement  $M(\mathcal{A}_n)$  in  $X^n$ . We restrict ourselves to types B, C, and D in this section because it allows us to work with one group – the hyperoctahedral group  $W_n$ . But the statements made here are true in type A if we consider  $S_n$  instead of  $W_n$ , by the work of Church [Chu12].

We may consider the Leray spectral sequence of the inclusion  $f : M(\mathcal{A}_n) \hookrightarrow X^n$ , which is given by

$$E_2^{pq}(n) = H^p(X^n; R^q f_* \mathbb{Q}) \implies H^{p+q}(M(\mathcal{A}_n); \mathbb{Q}).$$

Our goal is to show representation stability of the cohomology, and so we start by understanding the  $E_2$ -term as a representation.

**Lemma 4.6.** *Let  $p, q \geq 0$  and  $n \geq p + 2q$ . Then for some finite indexing set  $I = \{(\lambda, r, \alpha)\}$ , there are  $W_k$ -representations  $V(\lambda, r, \alpha)$  (where  $k \leq p + 2q$  depends on  $(\lambda, r, \alpha)$ ) such that*

$$E_2^{pq}(n) = \bigoplus_I \mathrm{Ind}_{W_k \times W_{n-k}}^{W_n} V(\lambda, r, \alpha) \boxtimes \mathbb{Q}.$$

*Proof.* Fix  $p, q \geq 0$  and  $n \geq p + 2q$ . We start with a known decomposition [Bib16, Lemma 3.1] as follows, which we write using our description of components from Theorem 3.3:

$$E_2^{pq}(n) = \bigoplus_{\Sigma} H^p(F_{\Sigma}; \mathbb{Q}) \otimes H^q(M(\mathcal{A}_{F_{\Sigma}}); \mathbb{Q})$$

where the sum is taken over all  $\Sigma \in \mathcal{C}_n$  such that  $\text{rk}(\Sigma) = q$ . We recall that  $F_\Sigma$  denotes the component of  $\mathcal{A}_n$  corresponding to the partition  $\Sigma$  (as in the proof of Theorem 3.3), and  $\mathcal{A}_{F_\Sigma}$  denotes the localization of the arrangement  $\mathcal{A}_n$  at  $F_\Sigma$  (as in Section 2.1). From here on, cohomology is always with rational coefficients.

The action of  $W_n$  will permute the summands according to its action on the set  $\mathcal{C}_n$ . We have already noted the orbits of this action; the orbit of  $\Sigma$  is given by  $\{\Sigma' \in \mathcal{C}_n \mid \widehat{\Sigma'} = \widehat{\Sigma}\}$  where  $\widehat{\Sigma} \in \mathcal{Q}_n$  is a labelled partition of  $n$ . We will introduce a more convenient way (independent of  $n$ ) to index these orbits.

For  $\lambda \in \mathcal{Q}_q$ , define  $\lambda\langle n \rangle \in \mathcal{Q}_n$  as follows: say  $(\lambda\langle n \rangle)_z = \lambda_z$  and if  $(\lambda_1, \dots, \lambda_t)$  are the unlabelled parts of  $\lambda$  with  $\lambda_1 \geq \dots \geq \lambda_t > 0$ , let  $(\lambda_1 + 1, \dots, \lambda_t + 1, 1, \dots, 1)$  be the unlabelled parts of  $\lambda\langle n \rangle$ . In order for this to be a partition of  $n$ , note that we must add  $n - q - t$  1's to the partition. For example, if  $\lambda$  is the labelled partition  $(1_0, 2, 1)$  of 4, then  $\lambda\langle 8 \rangle$  is the partition  $(1_0, 3, 2, 1, 1)$ .

Now, for each  $\Sigma \in \mathcal{C}_n$  with rank  $q$ , there is some  $\lambda \in \mathcal{Q}_q$  such that  $\lambda\langle n \rangle = \widehat{\Sigma}$ . This means that we can rearrange our decomposition as follows:

$$E_2^{pq}(n) = \bigoplus_{\lambda \in \mathcal{Q}_q} \bigoplus_{\widehat{\Sigma} = \lambda\langle n \rangle} H^p(F_\Sigma) \otimes H^q(M(\mathcal{A}_{F_\Sigma})).$$

In the definition of  $F_\Sigma$ , for  $\Sigma \in \mathcal{C}_n$ , there are no conditions imposed on the coordinates of  $X^n$  corresponding to the singletons in  $\Sigma$ . That is,  $F_\Sigma$  breaks up into a product

$$F'_\Sigma \times X_{i_1} \times \dots \times X_{i_s},$$

where each subscript  $i_j$  denotes the coordinate in which the factor  $X$  appears, which corresponds to pair of singletons  $\{i_j\}, \{\bar{i}_j\} \in \Sigma$ . This means we can use the Künneth formula to write

$$H^p(F_\Sigma) = \bigoplus_{r + \sum a_i = p} H^r(F'_\Sigma) \otimes H^{a_1}(X_{i_1}) \otimes \dots \otimes H^{a_s}(X_{i_s}).$$

We denote  $a = (a_1, \dots, a_s)$ , and let  $\widehat{a} \vdash (p - r)$  be the partition which lists the nonzero elements of  $a$  in decreasing order. For example, if  $a = (0, 2, 0, 1, 2)$ , then  $\alpha = (2, 2, 1)$ .

Given  $\lambda \in \mathcal{Q}_q$ ,  $r \in \{0, \dots, p\}$ , and  $\alpha \vdash (p - r)$ , we will define the following:

$$E(\lambda, r, \alpha)_n = \bigoplus_{\widehat{\Sigma} = \lambda\langle n \rangle} \bigoplus_{\widehat{a} = \alpha} H^r(F'_\Sigma) \otimes H^{a_1}(X_{i_1}) \otimes \dots \otimes H^{a_s}(X_{i_s}) \otimes H^q(M(\mathcal{A}_{F_\Sigma})).$$

The action of  $W_n$  permutes the summands of  $E(\lambda, r, \alpha)_n$  according to its action on  $(\Sigma, a)$ , and so this gives us the decomposition of  $W_n$ -representations:

$$E_2^{pq}(n) = \bigoplus_{\lambda \in \mathcal{Q}_q} \bigoplus_{r=0}^p \bigoplus_{\alpha \vdash (p-r)} E(\lambda, r, \alpha)_n.$$

The triples  $(\lambda, r, \alpha)$  are the indexing set in the statement of the lemma. So it remains to show that  $E(\lambda, r, \alpha)_n$  is an induced representation.

To see this, we first observe that  $W_n$  acts transitively on the summands of  $E(\lambda, r, \alpha)_n$ . This implies that  $E(\lambda, r, \alpha)_n = \text{Ind}_G^{W_n} V(\Sigma, r, a)$ , where we denote an arbitrary summand of  $E(\lambda, r, \alpha)_n$  by  $V(\Sigma, r, a)$  and its stabilizer by  $G$ . Here, we must have  $\widehat{\Sigma} = \lambda\langle n \rangle$  and  $\widehat{a} = \alpha$ , and all summands have the same value of  $r$ , but we can pick a particularly nice choice of  $\Sigma$  and  $a$ . That is, we can “left-justify” in the same way that Church does, by taking  $a = (\alpha_1, \dots, \alpha_t, 0, \dots, 0)$  and  $\Sigma$  to have

singletons  $\{n - s + 1\}$ ,  $\{\overline{n - s + 1}\}$ ,  $\dots$ ,  $\{n\}$ ,  $\{\bar{n}\}$  along with some fixed partition of  $\mathbf{n-s}$  (independent of  $n$ ).

Since there are  $s$  pairs of singleton parts and  $\ell(\alpha) = t$ , we let  $k = n - s + t$ . Consider  $W_k$  as the subgroup of  $W_n$  which acts on  $\mathbf{k}$ , and consider  $W_{n-k}$  as acting on  $\mathbf{n} \setminus \mathbf{k}$ . The stabilizer  $G$  of our summand  $V(\Sigma, r, \alpha)$  satisfies  $W_{n-k} \subseteq G \subseteq W_k \times W_{n-k}$ , and moreover,  $W_{n-k}$  acts trivially on  $V(\Sigma, r, \alpha)$ . Thus, we can write  $G = H \times W_{n-k}$  for some  $H \subseteq W_k$  and view  $V(\Sigma, r, \alpha)$  as a representation over  $H$ . We define  $V(\lambda, r, \alpha)_n = \text{Ind}_H^{W_k} V(\Sigma, r, \alpha)$ . Note that by our choice of  $\Sigma$  and  $\alpha$ , the representation  $V(\lambda, r, \alpha)$  does not depend on  $n$ .

Finally,

$$\begin{aligned} E(\lambda, r, \alpha)_n &= \text{Ind}_G^{W_n} V(\Sigma, r, \alpha) \\ &= \text{Ind}_{H \times W_{n-k}}^{W_n} V(\Sigma, r, \alpha) \boxtimes \mathbb{Q} \\ &= \text{Ind}_{W_k \times W_{n-k}}^{W_n} \left( \text{Ind}_H^{W_k \times W_{n-k}} V(\Sigma, r, \alpha) \boxtimes \mathbb{Q} \right) \\ &= \text{Ind}_{W_k \times W_{n-k}}^{W_n} \left( \text{Ind}_H^{W_k} V(\Sigma, r, \alpha) \right) \boxtimes \mathbb{Q} \\ &= \text{Ind}_{W_k \times W_{n-k}}^{W_n} V(\lambda, r, \alpha) \boxtimes \mathbb{Q}. \end{aligned}$$

□

**Theorem 4.7.** *Let  $\{\mathcal{A}_n\}$  be a sequence of linear, toric, or elliptic arrangements in type B, C, or D. Then for each  $i \geq 0$ , the sequence  $\{H^i(M(\mathcal{A}_n); \mathbb{Q})\}$  of  $W_n$ -representations is uniformly representation stable with stable range  $n \geq 4i$ .*

*Proof.* Note that for any inclusion  $\iota : \mathbf{n} \hookrightarrow \mathbf{m}$  with  $\iota(\bar{k}) = \overline{\iota(k)}$ , we have induced maps  $X^m \rightarrow X^n$  and  $M(\mathcal{A}_m) \rightarrow M(\mathcal{A}_n)$ . By functoriality of the Leray spectral sequence, we have maps from the Leray spectral sequence associated to  $\mathcal{A}_n$  to that of  $\mathcal{A}_m$ . This makes the Leray spectral sequence a spectral sequence of  $\text{FI}_{\mathcal{W}}$ -modules. We claim that our decomposition of  $E_2^{pq}(n)$  in Lemma 4.6 actually gives us a decomposition of  $\text{FI}_{\mathcal{W}}$ -modules

$$E_2^{pq} = \bigoplus_I M_W(V(\lambda, r, \alpha)).$$

Since each summand on the right-hand side is finitely generated with weight  $\leq p+2q$  and stability degree  $\leq p+2q$ , the finite direct sum giving  $E_2^{pq}$  must be as well. The theorem then follows from Theorems 4.2 and 4.4.

To see our decomposition as  $\text{FI}_{\mathcal{W}}$ -modules, the fact that  $V(\lambda, r, \alpha)$  does not depend on  $n$  tells us that for each morphism (ie, injection)  $\iota : \mathbf{n} \rightarrow \mathbf{n+1}$  in the category  $\text{FI}_{\mathcal{W}}$ , the following diagram commutes:

$$\begin{array}{ccc} E_2^{pq}(n) & \longrightarrow & E_2^{pq}(n+1) \\ \updownarrow & & \updownarrow \\ \bigoplus \text{Ind}_{W_k \times W_{n-k}}^{W_n} V(\lambda, r, \alpha) \boxtimes \mathbb{Q} & \longrightarrow & \bigoplus \text{Ind}_{W_k \times W_{n+1-k}}^{W_{n+1}} V(\lambda, r, \alpha) \boxtimes \mathbb{Q} \end{array}$$

□

**Remark 4.8.** The argument in Lemma 4.6 is very similar to that given by Church [Chu12] in the type A case, generalized to work with  $W_n$  and labelled partitions (rather than  $S_n$  and partitions). In a separate paper, Church, Ellenberg, and Farb [CEF15, Theorem 6.2.1] provide an alternative proof of representation stability for the type A case, which uses the fact that the  $E_2$ -term is generated by the cohomology of the linear arrangement along with the cohomology of the ambient space. The fact that these generators are finitely generated FI-modules is enough to show that each piece of the  $E_2$ -term is. However, in the other cases, the lack of unimodularity makes it more complicated. We still have that the cohomology of the linear arrangement and the cohomology of the ambient space give finitely generated  $\text{FI}_{\mathcal{W}}$ -modules [Wil15]. However, these together are not enough to generate the  $E_2$ -term. Instead of dealing with these extra generators separately, we have decided to follow Church’s original argument more closely.

**Remark 4.9.** We also remark that the spectral sequence degenerates somewhat quickly for our spaces. In the linear and toric cases, it degenerates at  $E_2$ ; in the elliptic case it degenerates at the  $E_3$  term by a Hodge theory argument [Bib16].

Now, there are a few easy consequences of the stability. Wilson showed that restriction of  $\text{FI}_{\mathcal{W}}$ -modules preserves finite generation [Wil14, Prop. 3.22], which gives us the following.

**Corollary 4.10.** *In each case, the cohomology restricted to either the type A Weyl group ( $S_n$ ) or the type D Weyl group remains representation stable.*

We also get an analogue of [Wil15, Cor. 5.10] on the polynomiality of characters.

**Corollary 4.11.** *The sequence of characters of the  $W_n$ -representations  $H^i(M(\mathcal{A}_n))$  are given by a unique character polynomial of degree  $\leq 2i$ . In particular, we have that  $\dim H^i(M(\mathcal{A}_n); \mathbb{Q})$  is a polynomial in  $n$  of degree  $\leq 2i$ .*

Finally, since  $H^i(M(\mathcal{A}_n)/\mathcal{W}_n; \mathbb{Q}) \cong H^i(M(\mathcal{A}_n); \mathbb{Q})^{\mathcal{W}_n}$  and Theorem 4.7 implies stability of the multiplicity of the trivial representation, we can make a statement on homological stability.

**Corollary 4.12.** *Let  $\{\mathcal{A}_n\}$  be a sequence of linear, toric, or elliptic arrangements of type A, B, C, or D with appropriate Weyl group  $\mathcal{W}_n$ . Then the orbit spaces  $M(\mathcal{A}_n)/\mathcal{W}_n$  enjoy rational homological stability. That is, for each  $i$ ,  $H^i(M(\mathcal{A}_n)/\mathcal{W}_n; \mathbb{Q})$  does not depend on  $n$  for  $n \geq 4i$ .*

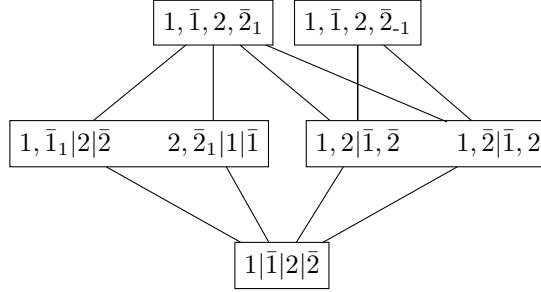
**4.4. Examples and Computations.** Computations such as finding the stable multiplicities of irreducible representations and the character polynomials are difficult in general. General computations of the stable multiplicities are not even known for the linear case or for the type A case. One aspect of the elliptic case that might make it harder is that not even the Betti numbers are known in general (there is a nice combinatorial description of the Betti numbers for linear and toric arrangements). But if one wanted to compute the stable multiplicities for the elliptic case, one might try to first compute them for the  $E_2$ -term. What you see, even in type A, is some tensor products of the linear case with an exterior algebra. Thus, even if the multiplicities of the linear case were known, one would have to deal with computation of the Kronecker coefficients from the tensor product.

We do show some work for the degree one cohomology. Even in degree two, though, it starts to get more complicated.

**Example 4.13.** Here are computations of the stable multiplicities in degree one cohomology of type A:

- (1) If  $X = \mathbb{C}$ , then  $H^1(M(\mathcal{A}_n)) = V(0) \oplus V(1) \oplus V(2)$  for  $n \geq 4$ . Church and Farb give this and a decomposition for degree two in [CF13].
- (2) If  $X = \mathbb{C}^\times$ , then  $H^1(M(\mathcal{A}_n)) = V(0)^{\oplus 2} \oplus V(1)^{\oplus 2} \oplus V(2)$  for  $n \geq 4$ .
- (3) If  $X$  is an elliptic curve, then we have  $E_\infty^{01}(n) = 0$  and hence for  $n \geq 2$ ,  $H^1(M(\mathcal{A}_n)) = E_2^{10}(n) = V(0)^{\oplus 2} \oplus V(1)^{\oplus 2}$ .

**Example 4.14.** In this example, we demonstrate the decomposition of Lemma 4.6 for  $H^1(M(\mathcal{A}_n); \mathbb{Q})$  in the case of a type  $B_2$  toric arrangement. Note that for  $X = \mathbb{C}^\times$ , the spectral sequence degenerates at the  $E_2$ -term, and so the decompositions of  $E_2^{01}$  and  $E_2^{10}$  together give a decomposition of the cohomology. First, recall our poset of labelled partitions. Here, we have grouped together the partitions which are in the same orbit.



Before we consider  $n = 2$ , note that for any  $n$ ,  $E_2^{10} = H^1((\mathbb{C}^\times)^n; \mathbb{Q})$ , which is the first degree part of an exterior algebra of  $\mathbb{Q}^n$ . The Weyl group acts in the standard way on  $\mathbb{Q}^n$ , giving us  $E_2^{10}(n) = V(n-1, 1)$ . This tells us that stably,  $E_2^{10}(n) = V(0, 1)_n$ , and in particular  $E_2^{10}(2) = V(1, 1)$ .

For  $E_2^{01}$ , we must have  $r = 0$  and  $\alpha = 0$ , and so our decomposition is indexed by the two orbits in the middle of the above picture. These correspond to the two labelled partitions of 1:  $(1_1)$  and  $(1)$ . The first orbit  $(1_1)$  gives us the induced representation of the  $W_1$ -representation  $V(1, \emptyset)$ . The second orbit gives us the  $W_2$ -representation  $\text{Ind}_{D_2}^{W_2} V$  where  $V = H^1(\mathbb{C}^2 \setminus H_{12}; \mathbb{Q}) = V(2, \emptyset)$  is considered as a representation over the stabilizer  $D_2$ . Thus,

$$H^1(M(\mathcal{A}_2); \mathbb{Q}) = V(1, 1) \bigoplus V(2, \emptyset) \bigoplus V(\emptyset, 2) \bigoplus \text{Ind}_{W_1 \times W_1}^{W_2} V(1, \emptyset) \boxtimes \mathbb{Q}.$$

Note that this does not give us the stable multiplicities. However, Wilson [Wil15] gave a decomposition of  $E_2^{01}$ , which we can consider as the first degree cohomology of the linear type B/C arrangement. This decomposition, as an  $\text{FI}_{BC}$ -module is  $M_{BC}(1, \emptyset) \oplus M_{BC}(2, \emptyset) \oplus M_{BC}(\emptyset, 2)$ . This, along with the fact that  $E_2^{10}(n) = V(0, 1)_n$ , tells us what  $H^1(M(\mathcal{A}_n); \mathbb{Q})$  should be stably.

If we had considered the type  $D_2$  arrangement, we would have

$$H^1(M(\mathcal{A}_2); \mathbb{Q}) = V(1, 1) \oplus V(2, \emptyset) \oplus V(\emptyset, 2).$$

The only difference from type  $B_2$  is that we have only one orbit of rank one, indexed by the partition  $(1)$ .

If we had considered the type  $C_2$  arrangement, we would have three orbits:  $(1_1)$ ,  $(1_{-1})$ , and  $(1)$ . The first and last would act as before; the new orbit would act just as  $(1_1)$  did. Thus, we would have the same decomposition as in the type  $B_2$  case with an extra factor of  $M_{BC}(1, \emptyset)$ .

**Example 4.15.** In this example, we demonstrate an aspect of the polynomiality of characters as in Corollary 4.11. Given a type of arrangement (a choice of  $X$  and the family of root systems), we state the dimension of  $H^1(M(\mathcal{A}_n); \mathbb{Q})$ . These formulas hold for all  $n \geq 2$ .

|                | $X = \mathbb{C}$    | $X = \mathbb{C}^\times$ | $X = E$             |
|----------------|---------------------|-------------------------|---------------------|
| Type $A_{n-1}$ | $\binom{n}{2}$      | $\binom{n}{2} + n$      | $2n$                |
| Type $B_n$     | $2\binom{n}{2} + n$ | $2\binom{n}{2} + 2n$    | $\binom{n}{2} + 2n$ |
| Type $C_n$     | $2\binom{n}{2} + n$ | $2\binom{n}{2} + 3n$    | $\binom{n}{2} + 5n$ |
| Type $D_n$     | $2\binom{n}{2}$     | $2\binom{n}{2} + n$     | $2n$                |

**4.5. An improvement for type A.** In our main results, we had ignored type A for ease of working only with  $W_n$  and because the result was already known, but the stable range for some type A arrangements can be improved. Recall that we know the sequence stabilizes once  $n \geq 4i$ , in each type (that is, for each  $X$  and family of root systems). Recently, Hersh and Reiner [HR15] improved the stable range for the type A linear case, showing that the  $i$ -th cohomology stabilizes for  $n \geq 3i + 1$ . We show an improvement for the elliptic case, and we wonder if it can be improved further, or if Hersh and Reiner's result can be used to improve the range of the toric case.

**Proposition 4.16.** *If  $\{\mathcal{A}_n\}$  is a sequence of type A elliptic arrangements, then for each  $i \geq 1$ , the stable range of  $\{H^i(M(\mathcal{A}_n); \mathbb{Q})\}$  may be improved to  $n \geq 4i - 2$ .*

*Proof.* Fix  $i \geq 1$ , and let  $\{\mathcal{A}_n\}$  be the sequence of type A elliptic arrangements. We claim that the differential  $d : E_2^{0,i}(n) \rightarrow E_2^{2,i-1}(n)$  is injective for all  $n$ , and hence  $E_\infty^{0,i} = 0$  for all  $n$ . Thus, in our filtration  $F_0 \subseteq \cdots \subseteq F_i = H^i$ , the maximum weight (and similarly stability degree) among  $F_j$  and  $F_j/F_{j-1}$  is  $2i - 1$ . This implies that  $H^i(M(\mathcal{A}_n); \mathbb{Q})$  is representation stable for  $n \geq 2(2i - 1) = 4i - 2$ .

To show injectivity of this differential, we pick our standard generators  $x_i, y_i$  ( $1 \leq i \leq n$ ) for  $H^*(X^n)$  and  $g_{ij}$  ( $1 \leq i < j \leq n$ ) for

$$E_2^{0,1}(n) = \bigoplus_{1 \leq i < j \leq n} H^1(M(\mathcal{A}_{H_{ij}})).$$

The differential sends  $g_{ij}$  to the class of the diagonal  $H_{ij}$  in  $H^2(X^n)$ , which is given by  $(x_i - x_j)(y_i - y_j)$ . A basis for  $E_2^{2,0} = H^2(X^n)$  is given by pairs  $x_i x_j, y_i y_j$  ( $i \neq j$ ) along with pairs  $x_i y_j$ . So if we had a linear combination  $\sum c_{ij} g_{ij}$  in the kernel of the differential, then its image  $\sum c_{ij} (x_i - x_j)(y_i - y_j)$  would be zero. We can write this sum in terms of the basis as  $\sum d_i x_i y_i - \sum c_{ij} (x_i y_j + x_j y_i)$  for some coefficients  $d_i$ . Since  $x_i y_j$  appears once, with coefficient  $c_{ij}$ , we must have  $c_{ij} = 0$ . The multiplication that exists on the  $E_2$ -term (higher degrees are generated by  $x_i$ 's,  $y_i$ 's, and  $g_{ij}$ 's) allows us to extend this argument to higher degrees.  $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN ONTARIO, LONDON, ON, CANADA  
*E-mail address:* cbibby2@uwo.ca